Deflated preconditioned conjugate gradient method for solving single-step single nucleotide polymorphism BLUP

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PROGRESS IN PIGS

Single-step models

- Prediction of genomic breeding values
  - Genotyped and non-genotyped animals
- Single-step GBLUP
  - Animal-based model
  - Limited?
- Single-step SNPBLUP
  - SNP-based model
  - Several equivalent formulations
  - No limitation?!

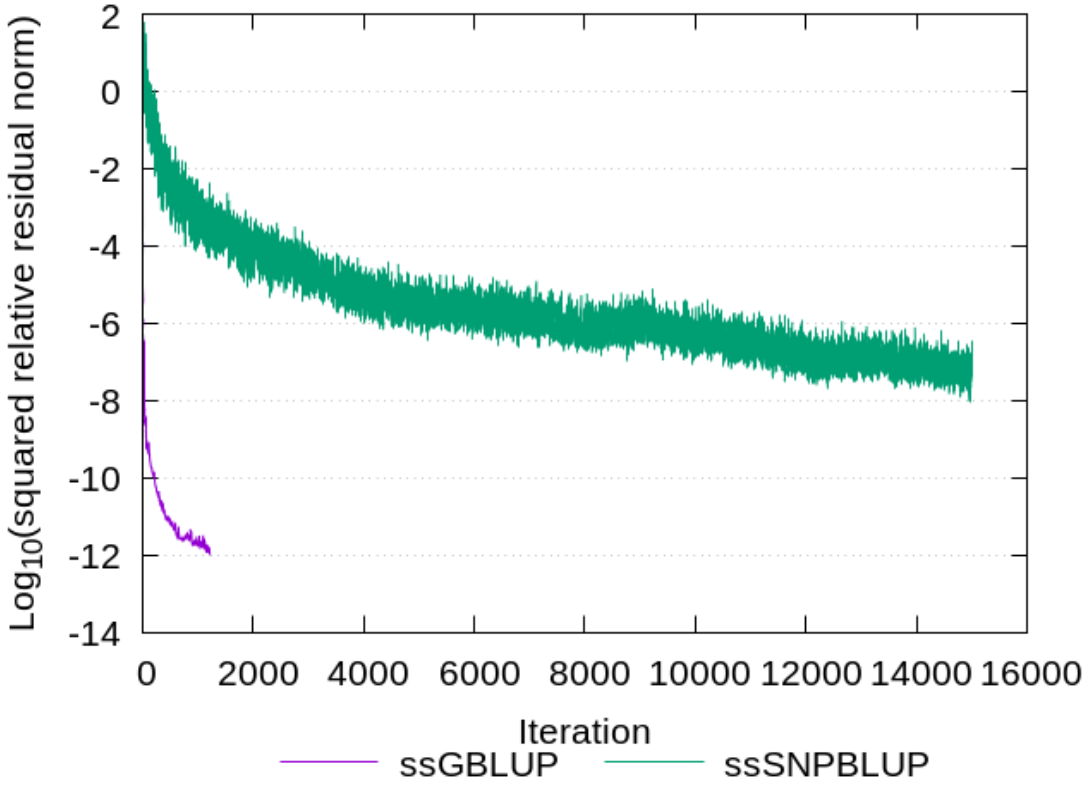
Fernando et al., 2016; Legarra et al., 2014; Liu et al., 2014; Taskinen et al., 2017

Single-step SNPBLUP

System of equations has the form **Cx = b**

➔Iterative solver: Preconditioned Conjugate Gradient

➔Convergence issues often reported...



Liu et al., 2014; Manzanilla Pech et al. 2017; Taskinen et al., 2017

Aim

1. Comparison of properties of coefficient matrices of ssGBLUP and ssSNPBLUP

➔Comprehension of convergence patterns of PCG

2. Implementation of a Deflated PCG method for solving efficiently ssSNPBLUP

Equivalent single-step models

ssGBLUP:
$$\mathbf{y} = \mathbf{Xb} + \begin{bmatrix} \mathbf{W}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_g \end{bmatrix} \begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_g \end{bmatrix} + \mathbf{e}$$

ssSNPBLUP:
$$\mathbf{y} = \mathbf{Xb} + \begin{bmatrix} \mathbf{W}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_g & \mathbf{W}_g\mathbf{Z} \end{bmatrix} \begin{bmatrix} \mathbf{u}_n \\ \mathbf{a}_g \\ \mathbf{g} \end{bmatrix} + \mathbf{e}$$

$$\mathbf{u}_g = \mathbf{a}_g + \mathbf{Zg}$$

Addition of SNP effects

$\mathbf{b}$: fixed effects

$\mathbf{u}_n$, $\mathbf{u}_g$: aggregate GEBVs for (non-)genotyped animals

$\mathbf{a}_g$: residual polygenic effects for genotyped animals

$\mathbf{g}$: SNP effects

$\mathbf{e}$: residuals

$\mathbf{Z}$: SNP genotypes

Fernando et al., 2016; Mantysaari and Stranden, 2016

Conjugate Gradient (CG)

- Successive approximations to obtain a more accurate solution of **x** by solving

$$\mathbf{Cx=b}$$

- Convergence
  - Function of the effective condition number of **C**

$$\kappa(\mathbf{C})=\frac{largest\ eigenvalue\ of\ \mathbf{C}}{(non-zero)\ smallest\ eigenvalue\ of\ \mathbf{C}}$$

  - Smaller condition number ➔ faster convergence

Preconditioned CG (PCG)

- Improvement of the condition number from $\kappa(\mathbf{C})$ to $\kappa(\mathbf{M}^{-1}\mathbf{C})$ by introducing a preconditioner **M**

$$\mathbf{M}^{-1}\mathbf{C}\mathbf{x}=\mathbf{M}^{-1}\mathbf{b}$$

- In animal breeding
  - PCG often implemented
  - Usually: **M** = *diag*(**C**) (or a variant)

Deflated PCG (DPCG)

- Improvement of the condition number from $\kappa(\mathbf{M}^{-1}\mathbf{C})$ to $\kappa(\mathbf{M}^{-1}\mathbf{PC})$ by introducing a second-level preconditioner **P**

$$\mathbf{M}^{-1}\mathbf{PCx} = \mathbf{M}^{-1}\mathbf{Pb}$$

P chosen such that unfavourable eigenvalues are set to 0 (deflated)

Deflated PCG (DPCG)

- **P** = deflation matrix

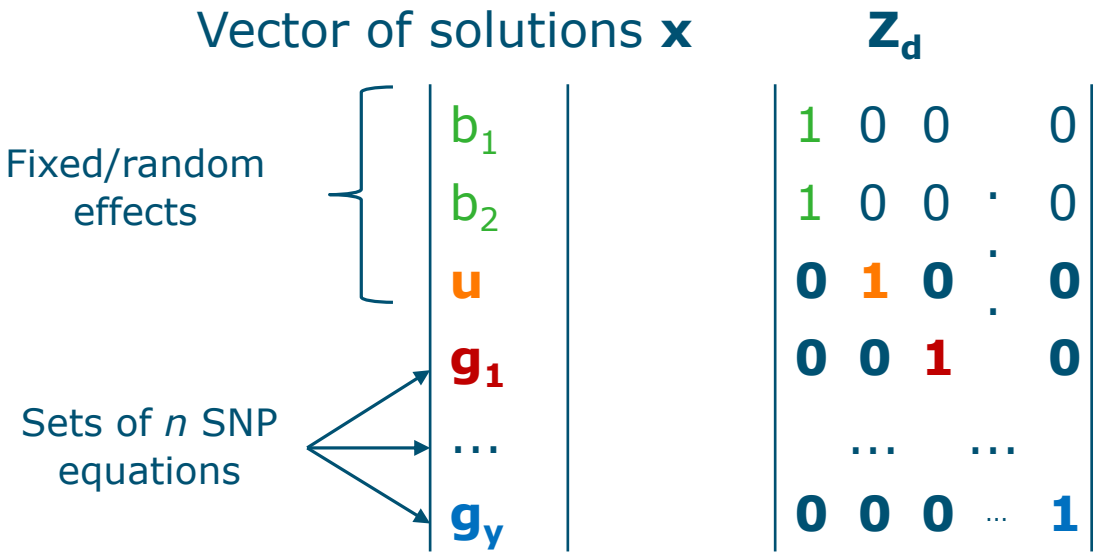
  $= \mathbf{I} - \mathbf{C}\mathbf{Z}_d(\mathbf{Z}_d'\mathbf{C}\mathbf{Z}_d)^{-1}\mathbf{Z}_d'$

- $\mathbf{Z}_d$ contains the deflation vectors
  - Approximation of the same space of the span of unfavourable eigenvectors

  ➔Set of deflation vectors ≈ Set of unfavourable eigenvectors

Deflated PCG (DPCG)

- Setting-up of $\mathbf{Z_d}$ following a subdomain deflation approach
  - 1 subdomain per fixed/random effect
  - 1 subdomain per set of *n* (1, 5, 50, or 200) SNP equations



Sparse $\mathbf{Z_d}$ ➔ efficient implementation ➔ small extra-costs

Data

- 61,592 Ovum pick-up sessions
- 37,021 animals
  - 4,109 phenotyped animals
  - 6,169 genotyped animals (without phenotype)
- 9,994 segregating SNPs

- Heritability = 0.35
- Residual polygenic variance = 5%

Cornelissen et al. 2017

Results – Spectra and condition numbers

Model	Method	Smallest eigenvalue	Largest eigenvalue	κ	#iter.
ssG	PCG	$1.1*10^{-4}$	11.9	$1.1*10^{5}$	273
ssSNP	PCG	$1.1*10^{-4}$	**181.0**	**$1.7*10^{6}$**	**1497**

ssSNPBLUP vs ssGBLUP

- Unchanged smallest eigenvalues
- Increased largest eigenvalues

➔Larger condition number

➔Increased number of iterations

Results – Spectra and condition numbers

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	DPCG (200)	$1.1*10^{-4}$	99.4	$9.3*10^{5}$	1195

200 SNPs equations per subdomain

➔Unchanged smallest value

➔Decreased largest eigenvalue

➔Better condition number after deflation

Results – Spectra and condition numbers

Model	Method	Smallest eigenvalue	Largest eigenvalue	κ	#iter.
ssG	PCG	$1.1*10^{-4}$	11.9	$1.1*10^{5}$	273
ssSNP	PCG	$1.1*10^{-4}$	**181.0**	**$1.7*10^{6}$**	**1497**
	DPCG (200)	$1.1*10^{-4}$	99.4	$9.3*10^{5}$	1195
	DPCG (50)	$1.1*10^{-4}$	40.5	$3.8*10^{5}$	880
	DPCG (5)	$1.1*10^{-4}$	**6.0**	**$5.6*10^{4}$**	**338**
	DPCG (1)	$1.1*10^{-4}$	**6.0**	**$5.4*10^{4}$**	**240**

1 and 5 SNPs per subdomain

- Similar (decreased) condition numbers
- #iterations similar as ssGBLUP
- Reduction of #iter. by up to a factor 6!

Conclusions

- ssSNPBLUP - PCG: larger eigenvalues
  - ➔Larger condition number
- ssSNPBLUP - Deflated PCG
  - Treats the largest unfavourable eigenvalues
  - ➔Smaller condition number
  - ➔Faster convergence (similar to ssGBLUP)
- Similar pattern on large and multivariate ssSNPBLUP

Thank you!

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